CH: 3 APPLICATIONS OF DERIVATIVES

★ **Slope of tangent (Gradient):**

For the curve $y = f(x)$, tangent is

$$f'(a) \text{ or } \left(\frac{dy}{dx}\right)_{\text{at } P} \text{ or } \left(\frac{dy}{dx}\right)_P$$

★ **Equation of tangent:**

$$(y-b) = f'(a)(x-a)$$

★ **Equation of normal:**

$$(y-b) = -\frac{1}{f'(a)}(x-a)$$

★ **Approximation:** If $f(x)$ is differentiable function of x, then approx. value of $f(a+h)$ is

$$f(a+h) \approx f(a) + h \cdot f'(a)$$

★ **Rolle's theorem:**

If $y = f(x)$ is a real valued function of a real variable such that

1) $f(x)$ is continuous on $[a,b]$
2) $f(x)$ is differentiable on (a,b)
3) $f(a) = f(b)$

then there exists a real number $c \in (a,b)$ such that $f'(c) = 0$.

★ **Lagrange's Mean-Value Theorem (LMVT):**

If $y = f(x)$ is a real valued function defined on $[a,b]$ such that

1) $f(x)$ is continuous on $[a,b]$
2) $f(x)$ is differentiable on (a,b) then

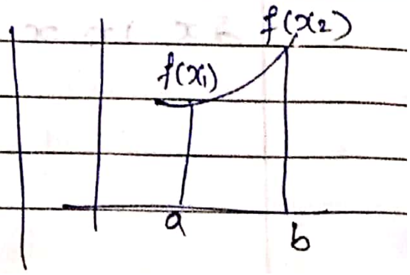
there exists at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

★ **Increasing function:**

$$f(x_2) > f(x_1)$$

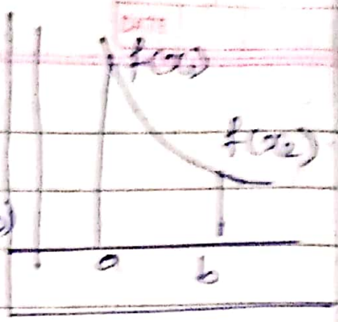
i.e. $f'(x) \geq 0$ for all $x \in (a,b)$.



* **Decreasing function:**

$f(x_2) < f(x_1)$

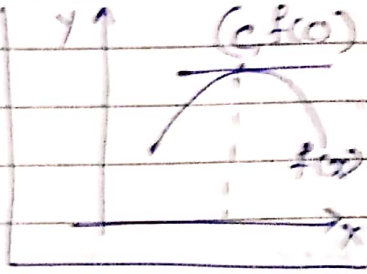
i.e. $f'(x) \leq 0$ for all $x \in (a,b)$



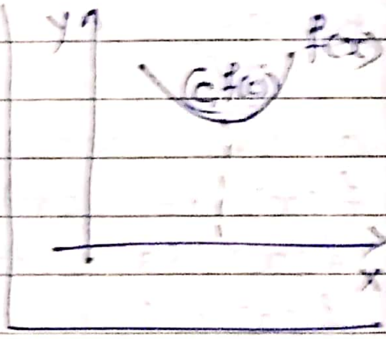
* **Maxima & Minima:**

1) **First derivative test:**

- A function $y=f(x)$ has maximum value at $x=c$ if
  ① $f'(c)=0$
  ② $f'(c-h)>0$
  ③ $f'(c+h)<0$ where h is small positive no.



- A function $y=f(x)$ has minimum value at $x=c$ if
  ① $f'(c)=0$
  ② $f'(c-h)<0$
  ③ $f'(c+h)>0$



2) **Second derivative test:**

A function $y=f(x)$ has maximum value at $x=c$ if

$f'(c)=0$ & $f''(c)<0$

& minimum value at $x=c$ if

$f'(c)=0$ & $f''(c)>0$

- If $f''(c)=0$ then second derivative test fails.

* **Error:** If $y=f(x)$ & δy is approx. error in y corresponding to very small error δx in x then $\delta y \approx \frac{dy}{dx} \cdot \delta x$

① Gradient of the curve $y = 2 + x + x^2$ at the point $x = -1$ on it is —

a) 0 b) -1 c) 2 d) 1

⇒ (b) $y = 2 + x + x^2$

$\therefore$ Slope $= \frac{dy}{dx} = 0 + 1 + 2x$

$\therefore \left.\frac{dy}{dx}\right|_{x=-1} = 1 + 2(-1)$

$= 1 - 2$

$= -1$

② Point at which the tangent to the curve $y = 3x - 2x^2$ has slope $m = -1$ is —

a) (0, 0) b) (1, 1) c) (-1, 1) d) (1, -1)

⇒ (b) $y = 3x - 2x^2$

$\therefore$ Slope $= \frac{dy}{dx} = 3 - 4x$

$\therefore 3 - 4x = -1$ ... (Given)

$\therefore 4x = 4$

$\therefore x = 1$

To find y-coordinate put $x = 1$ in eq^n.

For $x = 1$, $y = 3(1) - 2(1)^2$

$= 3 - 2$

$= 1$

$\therefore$ Point is (1, 1).

③ Slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $t = 2$ is ____

a) $\frac{7}{6}$ b) $\frac{5}{6}$ c) $\frac{6}{7}$ d) 1

⇒ (c) $x = t^2 + 3t - 8$

$\therefore \frac{dx}{dt} = 2t + 3$

$y = 2t^2 - 2t - 5$

$\therefore \frac{dy}{dt} = 4t - 2$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{4t-2}{2t+3}$$

$$\therefore \left.\frac{dy}{dx}\right|_{t=2} = \frac{4(2)-2}{2(2)+3} = \frac{8-2}{4+3} = \frac{6}{7}$$

(4) Inclination of the normal to the curve $xy = 15$ at the point $(3, 5)$ is ___

a) $\tan^{-1}\left(\frac{15}{9}\right)$ b) $-\tan^{-1}\left(\frac{9}{15}\right)$ c) $\tan^{-1}\left(\frac{9}{15}\right)$ d) $-\tan^{-1}\left(\frac{15}{9}\right)$

$\Rightarrow$ (c) $\quad xy = 15$

$$\therefore y = \frac{15}{x}$$

$$\therefore \text{Slope of tangent} = \frac{dy}{dx} = -\frac{15}{x^2}$$

$$\therefore \text{Slope of normal} = -\frac{1}{dy/dx} = \frac{x^2}{15}$$

$$\therefore \text{Inclination} = \tan\theta = \left.\frac{x^2}{15}\right|_{(3,5)}$$

$$\therefore \tan\theta = \frac{3^2}{15} = \frac{9}{15}$$

$$\therefore \theta = \tan^{-1}\left(\frac{9}{15}\right)$$

(5) Equations of the tangent & normal to the curve $x^5 + y^5 = 2xy$ at the point $(1, 1)$ are resp ___

a) $x - y = 2,\ x = y$ b) $x + y = 1,\ x = y$

c) $x + y = 2,\ x - y = 0$ d) $x - y = 2,\ x + y = 0$

$\Rightarrow$ (c) $\quad x^5 + y^5 = 2xy \quad ; P \equiv (1, 1)$

$$5x^4 + 5y^4 \cdot \frac{dy}{dx} = 2\left(x\frac{dy}{dx} + y\right)$$

$$\therefore 5(1)^4 + 5(1)^4 \cdot \frac{dy}{dx} = 2\left(1 \cdot \frac{dy}{dx} + 1\right)$$

$$\therefore 5 + 5 \cdot \frac{dy}{dx} = 2\frac{dy}{dx} + 2$$

$$\therefore (5-2)\frac{dy}{dx} = -3$$

$$\therefore \frac{dy}{dx} = \frac{-3}{3} = -1$$

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Equation of tangent

$y-1=-1(x-1)$

$y-1=-x+1$

$\therefore x+y=2$

Equation of normal

$y-1=1(x-1)$

$y-1=x-1$

$\therefore x-y=0$

⑥ Equation of tangent to the curve $4y=x^2+3x+2$, which is parallel to the x-axis is __

a) $16y+1=0$ b) $8y+1=0$

c) $4y+1=0$ d) $2y+1=0$.

$\Rightarrow$ (a) $4y=x^2+3x+2$ —①

Diff. $4\frac{dy}{dx}=2x+3$

$\therefore \frac{dy}{dx}=\frac{2x+3}{4}$

$\because$ Tangent is parallel to x-axis, $\therefore$ slope $=0$

$\therefore \frac{dy}{dx}=0$

$\therefore \frac{2x+3}{4}=0$ $\therefore 2x=-3$ $\therefore x=-3/2$

$\therefore$ Eqn ① $\Rightarrow$ $4y=\left(-\frac{3}{2}\right)^2+3\left(-\frac{3}{2}\right)+2$

$=\frac{9}{4}-\frac{9}{2}+2$

$=\frac{9-18+8}{4}=\frac{-1}{4}$

$\therefore y=-\frac{1}{16}$

$\therefore 16y=-1$

$\therefore 16y+1=0$ This is eqn of tangent.

⑦ If the line $y=4x-5$ touches the curve $y^2=ax^3+b$ at the point $(2,3)$ then $(a,b)\equiv$ __

a) $(2,-7)$ b) $(-2,7)$ c) $(2,7)$ d) $(-2,-7)$

$\Rightarrow$ (a) Curve: $y^2=ax^3+b$ Diff.

$\therefore 2y\cdot\frac{dy}{dx}=3ax^2$

$\therefore \frac{dy}{dx}=\frac{3ax^2}{2y}$

Tangent: $y=4x-5$

$\therefore 4x-y-5=0$

$\therefore$ slope $= \frac{-(4)}{-1} = 4$

$\therefore \left.\frac{dy}{dx}\right|_{(2,3)} = 4$

$\therefore \frac{3a(2)^2}{2(3)} = 4$

$\therefore \frac{12a}{6} = 4$

$\therefore 2a = 4 \qquad \therefore a = 2$

$\therefore$ Eq^n of curve $\quad y^2 = ax^3 + b$

$y^2 = 2(x^3) + b$

$\therefore (3)^2 = 2(2)^3 + b$

$\therefore 9 = 16 + b$

$\therefore b = -7$

$\therefore (a,b) \equiv (2,-7)$

H/W

(8) If the line $x+y=0$ touches the curve $2y^2 = ax^2 + b$ at the point $(1,-1)$ then $(a,b) \equiv$ ____

a) $(2,0)$ b) $(0,2)$ c) $(2,2)$ d) $(3,0)$

$\Rightarrow$ (a) $(2,0)$

(9) Distance of the normal to the curve $y = e^{2x} + x^2$, at the point $x=0$, from the origin is ___

a) $\frac{2}{\sqrt{3}}$ b) $\frac{2}{\sqrt{4}}$ c) $\frac{2}{\sqrt{5}}$ d) $\sqrt{2}$

$\Rightarrow$ (c) Curve: $y = e^{2x} + x^2$ — (1) Diff.

$\therefore \frac{dy}{dx} = 2e^{2x} + 2x$

$\therefore \left.\frac{dy}{dx}\right|_{x=0} = 2e^0 + 2(0)$

$= 2$

For $x=0$, eq^n (1) $\Rightarrow$

$y = e^0 + 0$

$\therefore y = 1$

$\therefore P \equiv (0,1)$, slope of normal $= -\frac{1}{2}$

$\therefore$ Equation of normal ; $y - 1 = -\frac{1}{2}(x - 0)$

$\therefore \cancel{y-1 = 2x}$

$\therefore 2y - 2 = -x$

$\therefore x + 2y - 2 = 0$

Distance of normal from origin is

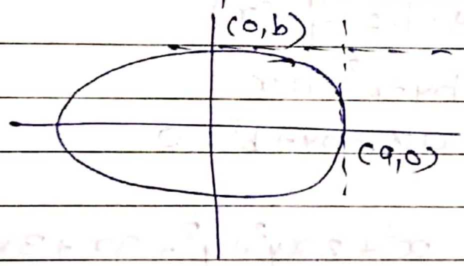
$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}} = \frac{|1(0) + 2(0) - 2|}{\sqrt{1^2+2^2}}$$

$$= \frac{|-2|}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

(10) The angle between the tangents to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points $(a,0)$ & $(0,b)$ is ___

a) $\frac{\pi}{4}$ b) $\frac{\pi}{2}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{6}$

$\Rightarrow$ (b) It is an ellipse. Draw sketch.



Thus the tangents at given points are at right angles. $\therefore$ Angle $= \pi/2$

(11) If $y = 4x - 5$ is tangent to the curve $y^2 = px^3 + q$ at $(2,3)$ then $(p,q) \equiv$ ___

a) $(2,-7)$ b) $(-2,7)$ c) $(-2,-7)$ d) $(2,7)$

$\Rightarrow$ (a) Curve $y^2 = px^3 + q$ —— (1) Diff.

$\therefore 2y \cdot \frac{dy}{dx} = 3px^2 + 0$

$\therefore \frac{dy}{dx} = \frac{3px^2}{2y}$

.... Same as Q. No. (7)

(12) Equation of normal to the curve $y = b \cdot e^{-x/a}$, where it cuts the y-axis is ___

a) $ax - by + b^2 = 0$ b) $ax - by + b = 0$

c) $ax - by + b^3 = 0$ d) $ax - by = 0$

⇒ (a) Curve: $y = b \cdot e^{-x/a}$ —①

It cuts y-axis ∴ $x = 0$

∴ Eqⁿ ① ⇒ $y = b \cdot e^0$

$\therefore y = b$

$\therefore P \equiv (0, b)$

Now $y = b \cdot e^{-x/a}$ Diff

$\therefore \frac{dy}{dx} = b \cdot e^{-x/a} \left(-\frac{1}{a}\right)$

$= -\frac{b}{a} \cdot e^{-x/a}$

$\therefore \left.\frac{dy}{dx}\right|_{(0,b)} = -\frac{b}{a} \cdot e^0 = -\frac{b}{a}$

∴ Equation of normal is ⇒

$y - b = \frac{-1}{-b/a}(x - 0)$

$\therefore y - b = \frac{a}{b}(x)$

$\therefore by - b^2 = ax$

$\therefore ax - by + b^2 = 0$

(13) The curve $x^4 + 2xy^2 + y^2 + 3x + 3y = 0$ cuts the x-axis at $(0, 0)$ at an angle of __

a) $\frac{\pi}{4}$ b) $\frac{\pi}{2}$ c) π d) $\frac{3\pi}{4}$

⇒ (d) Curve: $x^4 + 2xy^2 + y^2 + 3x + 3y = 0$

Diff.

$4x^3 + 2\left(x \cdot 2y \frac{dy}{dx} + y^2\right) + 2y \cdot \frac{dy}{dx} + 3 + 3 \cdot \frac{dy}{dx} = 0$

At $(0, 0)$

$\therefore 4(0) + 2(0 + 0) + 2(0) + 3 + 3 \cdot \frac{dy}{dx} = 0$

$\therefore 3 + 3 \cdot \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{3}{3} = -1$

$= -\tan 45^\circ$

$= -\tan(180^\circ - 135^\circ)$

$= \tan 135^\circ$

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$\therefore \theta = 135° = 135 \times \frac{\pi}{180} = \frac{3\pi^c}{4}$

(14) If $s = 9t - \frac{t^3}{3}$, then particle comes to rest after $t =$ ___

a) 1 sec b) 2 sec c) 3 sec d) 4 sec

$\Rightarrow$ (c) $s = 9t - \frac{t^3}{3}$

Particle comes to rest when $v = 0$ i.e. $\frac{ds}{dt} = 0$

i.e. $\frac{ds}{dt} = 9 - \frac{3t^2}{3}$

$\therefore 9 - t^2 = 0$

$\therefore t^2 = 9$

$\therefore t = 3$ sec

(15) If $s = t^3 - 6t^2 - 4t - 8$, then acceleration vanishes at time $t =$ ___

a) 2 sec b) 3 sec c) 4 sec d) 1 sec

$\Rightarrow$ (a) $s = t^3 - 6t^2 - 4t - 8$

Acceleration vanishes means $a = 0$

$\therefore \frac{ds}{dt} = 3t^2 - 12t - 4$

$\therefore \frac{d^2s}{dt^2} = 6t - 12$

$\therefore 6t - 12 = 0$ $(\because a = 0)$

$\therefore 6t = 12$

$\therefore t = 2$ sec

(16) If $s = t^3 - 6t^2 - 15t + 12$, then velocity at the time when acceleration becomes zero is ___

a) 15 units/sec b) -27 units/sec

c) 27 units/sec d) -15 units/sec

$\Rightarrow$ (b) $s = t^3 - 6t^2 - 15t + 12$

$\therefore v = \frac{ds}{dt} = 3t^2 - 12t - 15$ —— (1)

$\therefore a = \frac{dv}{dt} = 6t - 12$

At $a = 0 \Rightarrow \quad 6t - 12 = 0$

$\therefore 6t = 12$

$\therefore t = 2$

$\therefore Eq^n (1) \Rightarrow \quad v = 3(2)^2 - 12(2) - 15$

$= 12 - 24 - 15$

$= -27$ units/sec

(17) A stone thrown vertically upwards rises s ft in t seconds where $s = (112)t - (16)t^2$. The maximum height reached by the stone is —

a) 192 ft b) 190 ft c) 196 ft d) 392 ft

$\Rightarrow$ (c) $s = 112t - 16t^2$ —— (1)

$\therefore v = \frac{ds}{dt} = 112 - 32t$

At the maximum height, velocity of stone becomes zero.

$\therefore v = 0$

$\therefore 112 - 32t = 0$

$\therefore 32t = 112$

$\therefore t = \frac{112}{32} = \frac{7}{2}$

$Eq^n (1) \Rightarrow S_{max} = 112\left(\frac{7}{2}\right) - 16\left(\frac{7}{2}\right)^2$

$= 56 \times 7 - 4 \times 49$

$= 196$ ft

(18) The velocity of a particle moving in a straight line is proportional to the square-root of its displacement. Then the particle is moving with —

a) Variable acceleration

b) Acceleration proportional to velocity

c) Constant acceler^n d) Constant velocity

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$\Rightarrow$ (c) $v \propto \sqrt{s}$ ---- (Given)

$\therefore v = k \cdot \sqrt{s}$ —— (1)

$\therefore a = \frac{dv}{dt} = k \cdot \frac{1}{2\sqrt{s}} \cdot \frac{d}{dt}(s)$

$= \frac{k}{2\sqrt{s}} \cdot \frac{ds}{dt}$

$= \frac{k}{2\sqrt{s}} \cdot v$ $\quad \{ \because v = \frac{ds}{dt}$

$= \frac{k}{2\sqrt{s}} \cdot k\sqrt{s}$ ... From (1)

$= \frac{k^2}{2} =$ constant

$\therefore$ Particle is moving with constant acceln.

(19) Find approximate value of : $(2.01)^4 \approx$ ____

a) 16 b) 16.32 c) 16.5 d) 16.7

$\Rightarrow$ (b) Let $f(x) = x^4$ $\quad \therefore f'(x) = 4x^3$

$x = 2.01$ $\{$ ~~Let~~ : $a = 2$ & $h = 0.01$
$= 2 + 0.01$ $\therefore f(a) = f(2) = 2^4 = 16$

& $f'(a) = f'(2) = 4(2)^3 = 32$

$f(a+h) \approx f(a) + h \cdot f'(a)$

$\therefore (2.01)^4 \approx 16 + 0.01(32)$

$\approx 16 + 0.32$

≈ 16.32

(20) Find approximate value of : $(63)^{1/3} \approx$ ____

a) 3.2529 b) 3.7 c) 3.7892 d) 3.9972

$\Rightarrow$ (d) Let $f(x) = x^{1/3}$

$\therefore f'(x) = \frac{1}{3} x^{1/3 - 1} = \frac{1}{3 \cdot x^{2/3}} = \frac{1}{3(x^{1/3})^2}$

Let $a = 64$ & $h = -1$

$\therefore f(a) = f(64) = 64^{1/3} = (4^3)^{1/3} = 4$

& $f'(a) = f'(64) = \frac{1}{3(64^{1/3})^2} = \frac{1}{3[(4^3)^{1/3}]^2}$

$= \frac{1}{3(4)^2} = \frac{1}{3 \times 16} = \frac{1}{48} = 0.0208$

$f(a+h) \approx f(a) + h \cdot f'(a)$

$\therefore (63)^{1/3} \approx 4 + (-1)\, 0.0208$

$\approx 4 - 0.0208$

$\approx 3.9792 \qquad \approx 3.9972$

(21) Find approximate value of : $(1.0002)^{3000} \approx$ ____

a) 1.2   b) 1.4   c) 1.6   d) 1.8

$\Rightarrow$ (c) Let $f(x) = x^{3000}$

$\therefore f'(x) = 3000 \times x^{2999}$

Let $a = 1$ & $h = 0.0002$

$\cancel{f(a+h)} \therefore f(a) = f(1) = 1^{3000} = 1$

& $f'(a) = f'(1) = 3000 \times 1^{2999} = 3000$

$f(a+h) \approx f(a) + h \cdot f'(a)$

$\therefore (1.0002)^{3000} \approx 1 + (0.0002)\, 3000$

$\approx 1 + 0.6000$

≈ 1.6

(22) If $y = 4x^2 - 3x + 5$, $x = 4$, $\delta x = 0.04$ then $\delta y \approx$ ____

a) 0.116   b) 1.16   c) 1.1   d) 1.06

$\Rightarrow$ (b) $y = 4x^2 - 3x + 5$

$\therefore \frac{dy}{dx} = 8x - 3$

$\therefore \delta y \approx \frac{dy}{dx} \cdot \delta x$

$\approx (8x - 3) \cdot (0.04)$

$\approx [8(4) - 3](0.04) \qquad \{ \because x = 4 \ldots$ (Given)

$\approx (32 - 3)(0.04)$

$\approx 29 \times 0.04$

≈ 1.16